A unifying taxonomy of pattern matching in degenerate strings and founder graphs

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Degenerate Strings and Founder Graphs

$\Sigma = \{a, b\}$ alphabet.

Solid string: $ababaabbabbbabbaba$

Elastic-Degenerate (ED) string:

$$T = \underset{T[1]}{\begin{Bmatrix} aba \\ ba \end{Bmatrix}} \underset{T[2]}{\begin{Bmatrix} ba \\ b \end{Bmatrix}} \underset{T[3]}{\begin{Bmatrix} bb \\ aa \\ aba \end{Bmatrix}} \underset{T[4]}{\begin{Bmatrix} abbbaa \\ aabb \end{Bmatrix}} \underset{T[5]}{\begin{Bmatrix} b \\ ba \\ aba \end{Bmatrix}} \qquad \begin{aligned} n &= 5 \\ N &= 31 \end{aligned}$$

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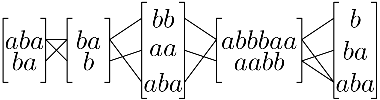
Elastic-Founder (EF) graph:

$G =$



$$\begin{aligned} n &= 5 \\ N &= 44 \\ |E| &= 13 \end{aligned}$$

Degenerate Strings and Founder Graphs

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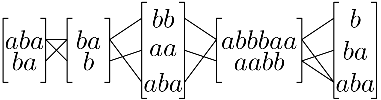


$$\begin{aligned} n &= 5 \\ N &= 44 \\ |E| &= 13 \end{aligned}$$

Variable string: degenerate string or founder graph (non solid).

Classification of degenerate strings

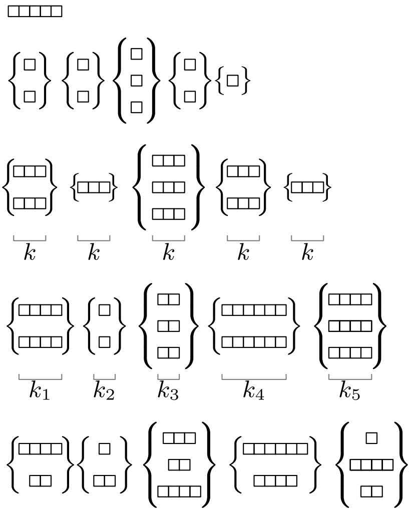
solid string

1-degenerate (1-D)

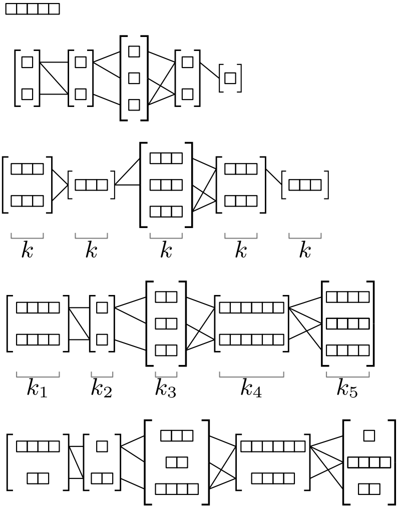
k-degenerate (k-D)

generalised degenerate (GD)

elastic degenerate (ED)



Classification of founder graphs



solid string

1-founder (1-F)

k-founder (k-F)

founder (F)

elastic founder (EF)

PVART(X, Y)

PATTERN MATCHING ON VARIABLE TEXT: PVART(X, Y)
Input: A solid or variable pattern P of type X and a variable text T of type Y
Output: All positions j such that an *occurence* of P in T ends at $T[j]$

Solid Pattern vs Solid Text: matching

$P = abbab$ $m = M = 5$

$T = ababaabbabbbabbaba$ $n = N = 18$

Solid Pattern vs Variable Text: matching

$$P = \quad abbab \qquad\qquad m = M = 5$$

$$T = \underbrace{\begin{Bmatrix} aba \\ ba \end{Bmatrix}}_{T[1]} \underbrace{\begin{Bmatrix} ba \\ b \end{Bmatrix}}_{T[2]} \underbrace{\begin{Bmatrix} bb \\ aa \\ aba \end{Bmatrix}}_{T[3]} \underbrace{\begin{Bmatrix} abbbaa \\ aabb \end{Bmatrix}}_{T[4]} \underbrace{\begin{Bmatrix} b \\ ba \\ aba \end{Bmatrix}}_{T[5]}$$

$$n = 5$$

$$N = 31$$

Solid Pattern vs Variable Text: matching

$$P = \; abbab \qquad\qquad m = M = 5$$

$$T = \; \underset{T[1]}{\begin{Bmatrix} aba \\ ba \end{Bmatrix}} \underset{T[2]}{\begin{Bmatrix} ba \\ b \end{Bmatrix}} \underset{T[3]}{\begin{Bmatrix} bb \\ aa \\ aba \end{Bmatrix}} \underset{T[4]}{\begin{Bmatrix} abbbaa \\ aabb \end{Bmatrix}} \underset{T[5]}{\begin{Bmatrix} b \\ ba \\ aba \end{Bmatrix}}$$

$$n = 5$$

$$N = 31$$

Variable Pattern vs Variable Text: matching

$$P = \left\{\begin{matrix} a \\ b \end{matrix}\right\} \{bba\} \left\{\begin{matrix} b \\ ab \end{matrix}\right\} \qquad \begin{matrix} m = 3 \\ M = 8 \end{matrix}$$

$$T = \left\{\begin{matrix} aba \\ ba \end{matrix}\right\}\left\{\begin{matrix} ba \\ b \end{matrix}\right\}\left\{\begin{matrix} bb \\ aa \\ aba \end{matrix}\right\}\left\{\begin{matrix} abbbaa \\ aabb \end{matrix}\right\}\left\{\begin{matrix} b \\ ba \\ aba \end{matrix}\right\} \qquad \begin{matrix} n = 5 \\ N = 31 \end{matrix}$$

Variable Pattern vs Variable Text: matching

$$P = \begin{Bmatrix} a \\ b \end{Bmatrix} \{bba\} \begin{Bmatrix} b \\ ab \end{Bmatrix} \qquad m = 3 \quad M = 8$$

$$\mathcal{L}(P) = abbab, bbbab, abbaab, bbbaab$$

$$T = \begin{Bmatrix} aba \\ ba \end{Bmatrix} \begin{Bmatrix} ba \\ b \end{Bmatrix} \begin{Bmatrix} bb \\ aa \\ aba \end{Bmatrix} \begin{Bmatrix} abbbaa \\ aabb \end{Bmatrix} \begin{Bmatrix} b \\ ba \\ aba \end{Bmatrix} \qquad n = 5 \quad N = 31$$

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$n = 5$

$N = 31$

Variable Pattern vs Variable Text: matching

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Main question

For each pair (X, Y), determine when $\textsc{PvarT}(\mathrm{X},\mathrm{Y})$ has:

a truly subquadratic upper bound

or

a quadratic lower bound.

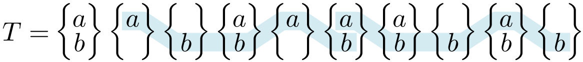
quadratic: $\mathcal{O}(NM)$
truly subquadratic: $\mathcal{O}(N \log M), \mathcal{O}(N\sqrt{M})$

$\textsc{PvarT}(\textsc{solid},1\text{-}D)$

Solid Pattern vs 1-D Text

1-D strings → *indeterminate* strings

$$P = abbab$$



$$T = \left\{\begin{matrix}a\\b\end{matrix}\right\}\left\{\begin{matrix}a\\ \end{matrix}\right\}\left\{\begin{matrix} \\b\end{matrix}\right\}\left\{\begin{matrix}a\\b\end{matrix}\right\}\left\{\begin{matrix}a\\ \end{matrix}\right\}\left\{\begin{matrix}a\\b\end{matrix}\right\}\left\{\begin{matrix}a\\b\end{matrix}\right\}\left\{\begin{matrix} \\b\end{matrix}\right\}\left\{\begin{matrix}a\\b\end{matrix}\right\}\left\{\begin{matrix} \\b\end{matrix}\right\}$$

$m = 5$, $n = 10$

Theorem (Cole and Hariharan 2002, [CH02])

$\textsc{PvarT}(\textsc{solid},1\text{-}D)$ can be solved in $\mathcal{O}(n \log^2 m)$ time.

PVART(SOLID,k-D)

Solid Pattern vs k-D Text

$$P = abbab$$

$$T = \underbrace{\begin{Bmatrix} abb \\ bab \end{Bmatrix}}_{k} \underbrace{\{bba\}}_{k} \underbrace{\begin{Bmatrix} bba \\ aab \\ abb \end{Bmatrix}}_{k} \underbrace{\begin{Bmatrix} baa \\ aab \end{Bmatrix}}_{k} \underbrace{\begin{Bmatrix} abb \\ bba \\ aaa \end{Bmatrix}}_{k}$$

$m = 5$, $n = 5$, $k = 3$,
$N = k * (\text{number of strings}) = 3 * 11 = 33$

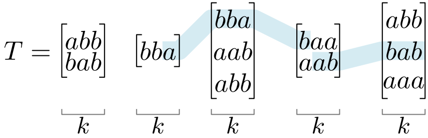
Theorem 1

PVART(SOLID,k-D) can be solved in $\mathcal{O}(N + N \log^2 m)$ time.

PVART(SOLID,k-F)

Solid Pattern vs k-F Text

$$P = abbab$$



$$T = \underbrace{\begin{bmatrix}abb\\bab\end{bmatrix}}_{k}\quad\underbrace{\begin{bmatrix}bba\end{bmatrix}}_{k}\quad\underbrace{\begin{bmatrix}bba\\aab\\abb\end{bmatrix}}_{k}\quad\underbrace{\begin{bmatrix}baa\\aab\end{bmatrix}}_{k}\quad\underbrace{\begin{bmatrix}abb\\bab\\aaa\end{bmatrix}}_{k}$$

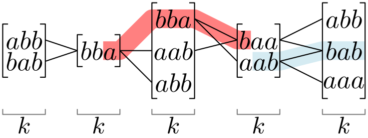
$|E|$ = number of edges = 11

PVART(SOLID,k-F)

Solid Pattern vs k-F Text

$P = abbab$

$T =$



$|E| =$ number of edges $= 11$

Theorem 2

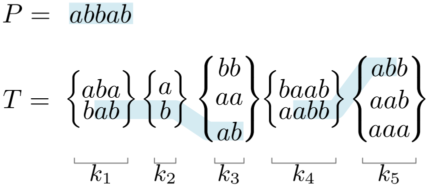
PVART(SOLID,k-F) can be solved in $\mathcal{O}(\sqrt{m}(|E| + N\log^2 m))$ time.

1-F
k-F
F
EF

PVART(SOLID,GD)

Solid Pattern vs GD Text

Generalized Degenerate (GD) Text



$$P = abbab$$

$$T = \underbrace{\begin{Bmatrix} aba \\ bab \end{Bmatrix}}_{k_1} \underbrace{\begin{Bmatrix} a \\ b \end{Bmatrix}}_{k_2} \underbrace{\begin{Bmatrix} bb \\ aa \\ ab \end{Bmatrix}}_{k_3} \underbrace{\begin{Bmatrix} baab \\ aabb \end{Bmatrix}}_{k_4} \underbrace{\begin{Bmatrix} abb \\ aab \\ aaa \end{Bmatrix}}_{k_5}$$

Theorem 3

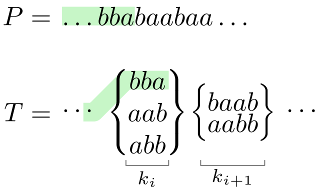
PVART(SOLID,GD) can be solved in $\mathcal{O}(N + nm)$ time.

PVART(SOLID,GD)

Key observation

Theorem 3

PVART(SOLID,GD) can be solved in $\mathcal{O}(N + nm)$ time.



$$P = \dots bbabaabaa \dots$$

$$T = \cdots \underbrace{\left\{\begin{matrix} bba \\ aab \\ abb \end{matrix}\right\}}_{k_i} \underbrace{\left\{\begin{matrix} baab \\ aabb \end{matrix}\right\}}_{k_{i+1}} \cdots$$

1-D

k-D

GD

ED

PVART(SOLID,GD)

Key observation

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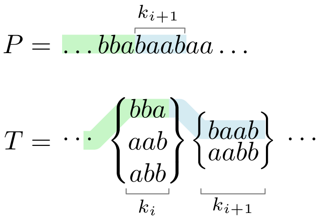
PVART(SOLID,GD) can be solved in $\mathcal{O}(N + nm)$ time.



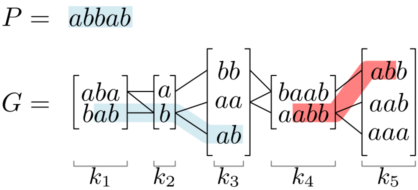
$$P = \dots bba\overbrace{baab}^{k_{i+1}}aa\dots$$

$$T = \cdots \underbrace{\begin{Bmatrix} bba \\ aab \\ abb \end{Bmatrix}}_{k_i} \underbrace{\begin{Bmatrix} baab \\ aabb \end{Bmatrix}}_{k_{i+1}} \cdots$$

$\textsc{PVarT}(\textsc{solid},F)$

Solid Pattern vs F Text

Founder (F) graph Text

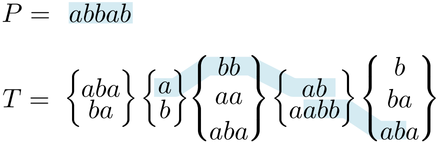


Theorem 4

$\textsc{PVarT}(\textsc{solid},F)$ can be solved in $\mathcal{O}(nm + N + |E|\log m)$ time.

PVART(SOLID,ED)

Solid Pattern vs ED Text



$$P = abbab$$

$$T = \begin{Bmatrix} aba \\ ba \end{Bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} \begin{Bmatrix} bb \\ aa \\ aba \end{Bmatrix} \begin{Bmatrix} ab \\ aabb \end{Bmatrix} \begin{Bmatrix} b \\ ba \\ aba \end{Bmatrix}$$

Theorem (Backurs and Indyk 2016, [BI16])

No algorithm can solve PVART(SOLID,ED) on constant alphabet in $\mathcal{O}(m^{1-\epsilon}N)$ nor in $\mathcal{O}(mN^{1-\epsilon})$ time for $\epsilon > 0$, unless OVH is false.

1-D

k-D

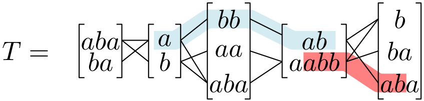
GD

ED

$\text{PVART}(\text{SOLID}, EF)$

Solid Pattern vs EF Text

$P = abbab$



$T = \begin{bmatrix} aba \\ ba \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} bb \\ aa \\ aba \end{bmatrix} \begin{bmatrix} ab \\ aabb \end{bmatrix} \begin{bmatrix} b \\ ba \\ aba \end{bmatrix}$

Theorem (Equi et al. 2021, [Equ+21])

No algorithm can solve $\text{PVART}(\text{SOLID}, EF)$ on constant alphabet in $\mathcal{O}(m^{1-\epsilon}|E|)$ nor in $\mathcal{O}(m|E|^{1-\epsilon})$ time for $\epsilon > 0$, unless OVH is false.

Summary tables: SOLID pattern

Degenerate strings

T P	1-D	k-D	GD	ED
SOLID	$\mathcal{O}(n \log^2 m)$	$\mathcal{O}(N \log^2 m)$	$\mathcal{O}(nm + N)$	$\Omega((mN)^{1-\epsilon})$

Founder graphs

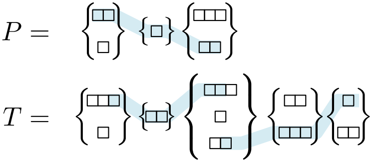
T P	1-F	k-F	F	EF
SOLID	$\mathcal{O}(\sqrt{m}(\lvert E\rvert + N \log^2 m))$		$\mathcal{O}(nm + N+$ $+\lvert E\rvert \log m)$	$\Omega((m\lvert E\rvert)^{1-\epsilon})$

Variable pattern: Degenerate vs Degenerate

Degenerate strings

Degenerate strings

P \ T	1-D	k-D	GD	ED
1-D				
k-D				
GD				
ED				



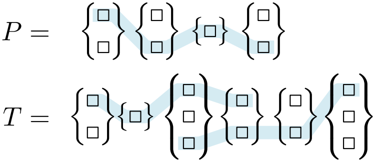
Variable pattern: PVART(1-D,1-D)

Degenerate strings

Degenerate strings

P \ T	1-D	k-D	GD	ED
1-D	[IR16]			
k-D				
GD				
ED				

[IR16] Iliopoulos and Radoszewski (CPM 2016)



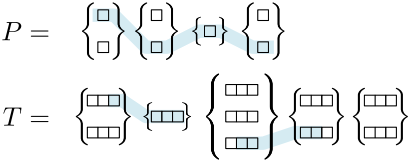
Variable pattern: PVART(1-D,k-D)

Degenerate strings

Degenerate strings

P \ T	1-D	k-D	GD	ED
1-D	[IR16]	Thm. 5		
k-D				
GD				
ED				

[IR16] Iliopoulos and Radoszewski (CPM 2016)



Orthogonal Vectors

ORTHOGONAL VECTORS (OV)
Given: sets $X, Y \subseteq \{0,1\}^d$ of size n such that $d = \omega(\log n)$
Determine: whether there exist $x \in X$ and $y \in Y$ such that x and y are orthogonal, i.e. $x \cdot y = \sum_{i=1}^{d} x[i] \cdot y[i] = 0$.

Example $n = 4, d = 3$:

$$X = \{\overbrace{010}^{x_1}, 100, 011, 111\}$$
$$Y = \{\underbrace{001}_{y_1}, 010, \underbrace{110}_{y_3}, 101\}$$

$x_1 \cdot y_1 = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0 \quad \Rightarrow$ orthogonal

$x_1 \cdot y_3 = 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 1 \quad \Rightarrow$ not orthogonal

Orthogonal Vectors Hypothesis

ORTHOGONAL VECTORS (OV)
Given: sets $X, Y \subseteq \{0,1\}^d$ of size n such that $d = \omega(\log n)$
Determine: whether there exist $x \in X$ and $y \in Y$ such that x and y are orthogonal, i.e. $x \cdot y = \sum_{i=1}^{d} x[i] \cdot y[i] = 0$.

Orthogonal Vectors Hypothesis (OVH)

For no $\epsilon > 0$, no algorithm can solve OV in time

$$\mathcal{O}(n^{2-\epsilon}\text{poly}(d)).$$

$\textsc{PvarT}(1\text{-}D, k\text{-}D)$

Theorem 5

No algorithm can solve $\textsc{PvarT}(1\text{-}D{,}k\text{-}D)$ on constant alphabet in $\mathcal{O}(M^{1-\epsilon}N)$ nor in $\mathcal{O}(MN^{1-\epsilon})$ time for $\epsilon > 0$, unless OVH is false.

PVART(1-D, k-D)

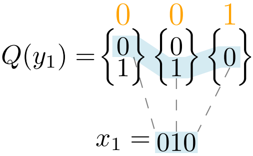
Theorem 5

No algorithm can solve PVART(1-D,k-D) on constant alphabet in $\mathcal{O}(M^{1-\epsilon}N)$ nor in $\mathcal{O}(MN^{1-\epsilon})$ time for $\epsilon > 0$, unless OVH is false.

$X = \{010, 100, 011\}$
$Y = \{001, 010, 110\}$

$$y_1 = 001 \quad \Rightarrow \quad Q(y_1) = \overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}} \overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}} \overset{1}{\begin{Bmatrix}0\end{Bmatrix}}$$

x_1 matches $Q(y_1)$ if and only if x_1 and y_1 are orthogonal:



PVART(1-D,k-D)

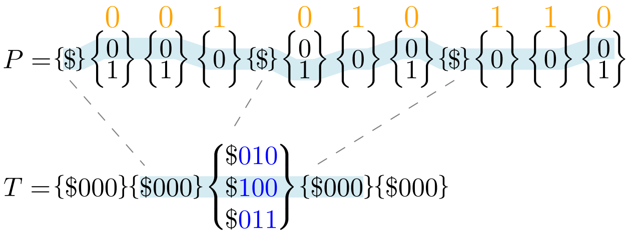
$$X = \{010, 100, 011\}$$
$$Y = \{001, 010, 110\}$$

$$P = \{\$\}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\{\$\}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\{\$\}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}$$

$$T = \{\$000\}\{\$000\}\begin{Bmatrix}\$010\\ \$100\\ \$011\end{Bmatrix}\{\$000\}\{\$000\}$$

PvarT(1-D,k-D)

$$X = \{010, 100, 011\}$$
$$Y = \{001, 010, 110\}$$



$$P = \{\$\}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\{\$\}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}\{\$\}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{1}{\begin{Bmatrix}0\end{Bmatrix}}\overset{0}{\begin{Bmatrix}0\\1\end{Bmatrix}}$$

$$T = \{\$000\}\{\$000\}\begin{Bmatrix}\$010\\\$100\\\$011\end{Bmatrix}\{\$000\}\{\$000\}$$

PvarT(1-D,k-D)

X, $Y \subseteq \{0,1\}^d$ of size n

size of P: $M = \mathcal{O}(nd)$

size of T: $N = \mathcal{O}(nd)$

time to build P and T: $\mathcal{O}(nd)$

$\textsc{PvarT}(1\text{-}D,k\text{-}D)$

X, $Y \subseteq \{0,1\}^d$ of size n

size of P: $M = \mathcal{O}(nd)$

size of T: $N = \mathcal{O}(nd)$

time to build P and T: $\mathcal{O}(nd)$

Given an $\mathcal{O}(MN^{1-\epsilon})$-time algorithm for $\textsc{PvarT}(1\text{-}D,k\text{-}D)$

$$\Downarrow$$

we get an algorithm for OV running in time

$$\overbrace{\mathcal{O}(nd)}^{\text{build } P \text{ and } T} \quad + \quad \overbrace{\mathcal{O}((nd)(nd)^{1-\epsilon})}^{\textsc{PvarT} \text{ algorithm}} = \mathcal{O}(n^{2-\epsilon}\text{poly}(d))$$

which contraddicts OVH.

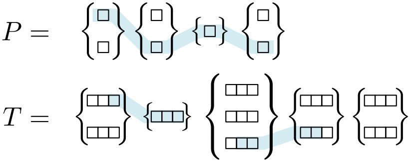
Variable pattern:

Degenerate strings

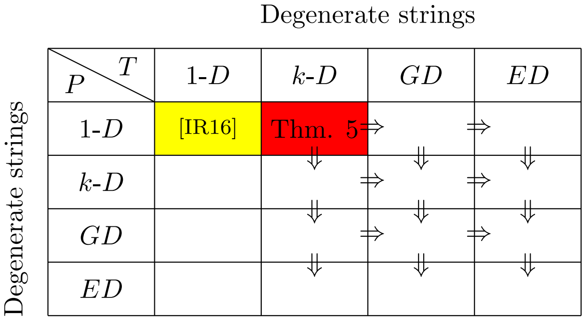
Degenerate strings

T P	1-D	k-D	GD	ED
1-D	[IR16]	Thm. 5		
k-D				
GD				
ED				

[IR16] Iliopoulos and Radoszewski (CPM 2016)



Variable pattern: $\textsc{PvarT}(1\text{-}D, k\text{-}D)$

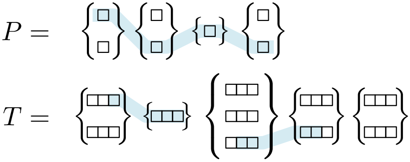


Degenerate strings

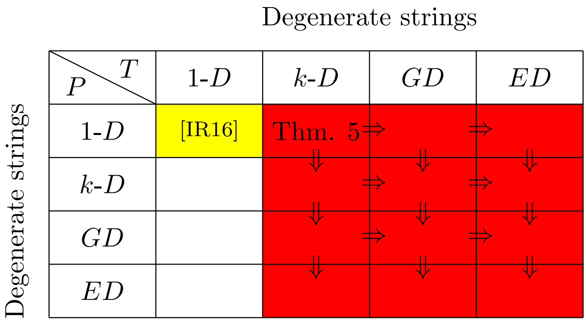
P \ T	1-D	k-D	GD	ED
1-D	[IR16]	Thm. 5	⇒	⇒
k-D		⇓	⇓ ⇒	⇓ ⇒
GD		⇓	⇓ ⇒	⇓ ⇒
ED		⇓	⇓	⇓

Degenerate strings

[IR16] Iliopoulos and Radoszewski (CPM 2016)



Variable pattern: PVART(1-D,k-D)

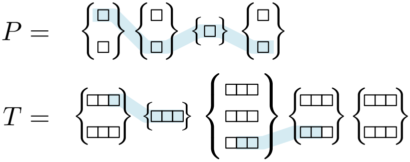


Degenerate strings

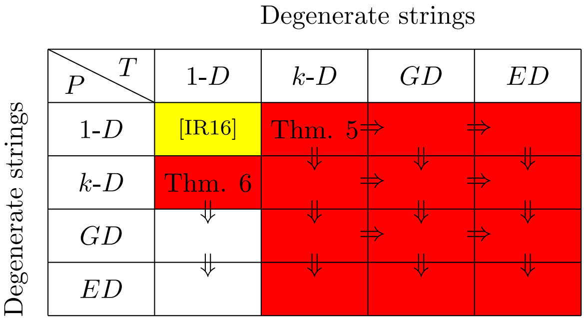
P \ T	1-D	k-D	GD	ED
1-D	[IR16]	Thm. 5	⇒	⇒
k-D		⇓	⇒ ⇓	⇒ ⇓
GD		⇓	⇒ ⇓	⇒ ⇓
ED		⇓	⇓	⇓

Degenerate strings

[IR16] Iliopoulos and Radoszewski (CPM 2016)



Variable pattern: PVART(k-D,1-D)

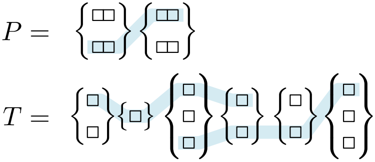


Degenerate strings

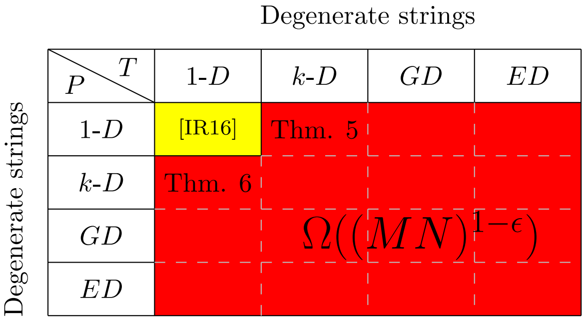
Degenerate strings

P \ T	1-D	k-D	GD	ED
1-D	🟨 [IR16]	🟥 Thm. 5⇒	🟥 ⇒	🟥
k-D	🟥 Thm. 6	🟥 ⇓ ⇒	🟥 ⇓ ⇒	🟥 ⇓
GD	⇓	🟥 ⇓ ⇒	🟥 ⇓ ⇒	🟥 ⇓
ED	⇓	🟥 ⇓	🟥 ⇓	🟥 ⇓

[IR16] Iliopoulos and Radoszewski (CPM 2016)



Variable pattern: Degenerate vs Degenerate

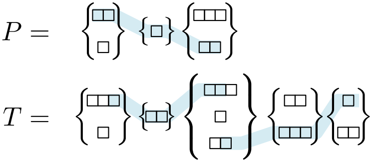


Degenerate strings

Degenerate strings

P \ T	1-D	k-D	GD	ED
1-D	[IR16]	Thm. 5		
k-D	Thm. 6			
GD		$\Omega((MN)^{1-\epsilon})$		
ED				

[IR16] Iliopoulos and Radoszewski (CPM 2016)



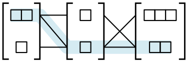
Variable pattern: Founder vs Degenerate

Degenerate strings

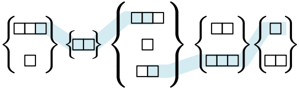
Founder graphs

T P	1-D	k-D	GD	ED
1-F				
k-F				
F				
EF				

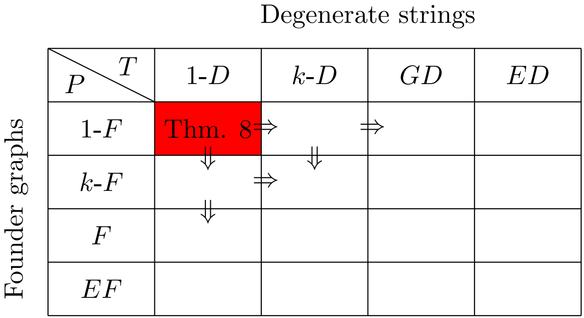
$P =$



$T =$



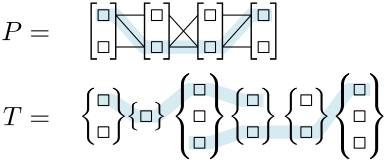
Variable pattern: PVarT(1-F,1-D)



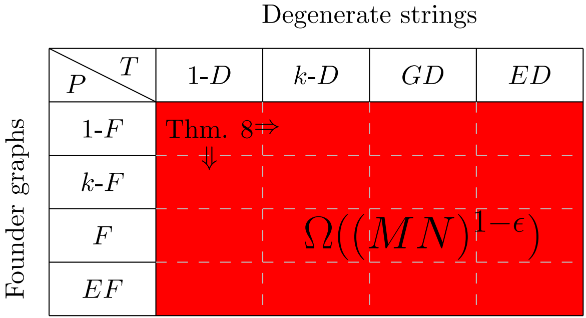
Degenerate strings

Founder graphs

P \ T	1-D	k-D	GD	ED
1-F	Thm. 8 ⇒ ⇓	⇓ ⇒		
k-F	⇒ ⇓			
F				
EF				



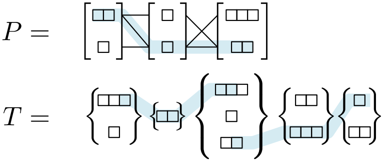
Variable pattern: Founder vs Degenerate



Degenerate strings

Founder graphs

P \ T	1-D	k-D	GD	ED
1-F	Thm. 8$\Rightarrow$			
k-F	$\Downarrow$			
F		$\Omega((MN)^{1-\epsilon})$		
EF				

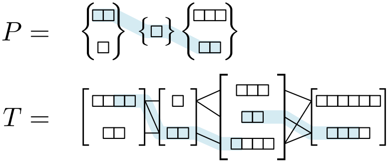


Variable pattern: Degenerate vs Founder

Founder graphs

Degenerate strings

T P	1-F	k-F	F	EF
1-D				
k-D				
GD				
ED				

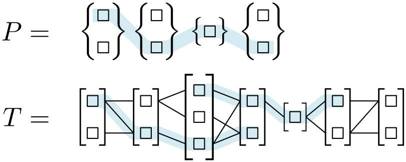


Variable pattern: PVART(1-D,1-F)

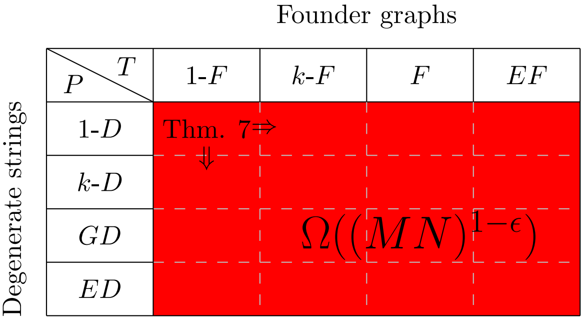
Founder graphs

Degenerate strings

P \ T	1-F	k-F	F	EF
1-D	Thm. 7⇒ ⇓			
k-D				
GD				
ED				



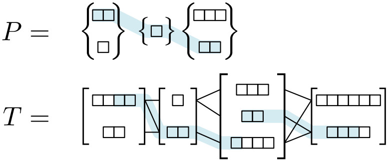
Variable pattern: Degenerate vs Founder



Founder graphs

Degenerate strings

P \ T	1-F	k-F	F	EF
1-D	Thm. 7⇒	🟥	🟥	🟥
k-D	⇓	🟥	🟥	🟥
GD	🟥	$\Omega((MN)^{1-\epsilon})$	🟥	🟥
ED	🟥	🟥	🟥	🟥



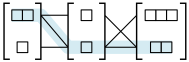
Variable pattern: Founder vs Founder

Founder graphs

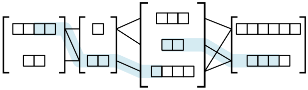
Founder graphs

T P	1-F	k-F	F	EF
1-F				
k-F				
F				
EF				

$P =$



$T =$



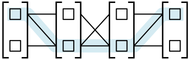
Variable pattern: PVART(1-F,1-F)

Founder graphs

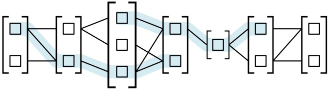
Founder graphs

T P	1-F	k-F	F	EF
1-F	🟥 Thm. 9 ⇒ ⇓			
k-F				
F				
EF				

$P =$

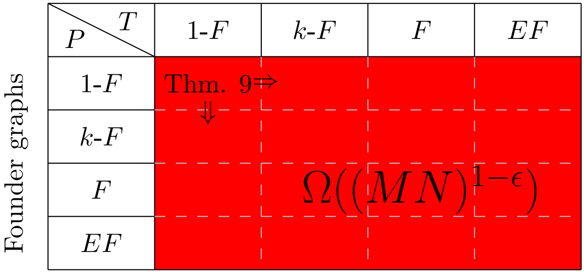


$T =$



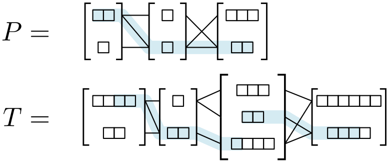
Variable pattern: Founder vs Founder

Founder graphs



P \ T	1-F	k-F	F	EF
1-F	Thm. 9⇒			
k-F	⇓			
F		$\Omega((MN)^{1-\epsilon})$		
EF				

Founder graphs

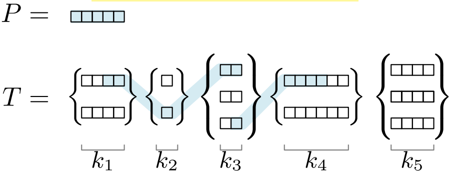


Conclusions

- $\textsc{PvarT}(\textsc{solid}, k\text{-}D)$: truly subquadratic upper bound
- $\textsc{PvarT}(\textsc{solid}, k\text{-}F)$: truly subquadratic upper bound
- If at least one between pattern and text is a founder and the other is variable string: quadratic conditional lower bound (even for constant-size alphabets).

Open problems

$\text{PVarT}(\text{solid}, GD)$



$\text{PVarT}(\text{solid}, F)$